PERIODIC CONVOLUTION EQUATIONS ON SEGMENTS WHICH OCCUR IN ELASTICITY THEORY AND MATHEMATICAL PHYSICS

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Integral convolution equations given on a system of segments are considered; the kernel of the integral equation is a periodic function with period T The unique solvability of the equations is established for kernels of a general kind, an approximate factorization is given a foundation, and a method for construction of the approximate solution is also indicated.

Integral equations of the kind mentioned occur in mixed problems of elasticity theory and mathematical physics, posed for finite bodies [1, 2], as well as for infinite bodies with periodic boundary conditions [3]. The result herein, just as in [4], is necessary for the correct formulation and solution of dynamic contact problems.

1. Let us consider the integral equation

$$\mathbf{K}_q \equiv \sum_{k=1}^{N} \int\limits_{a_{2k-1}}^{a_{2k}} k(x-\xi)\, q_{2k-1}(\xi)\, d\xi = 2\pi f_{2m-1}(x) \equiv 2\pi f \tag{1.1}$$

$$a_{2m-1} \leqslant x \leqslant a_{2m}, \qquad a_{2N} - a_1 < T, \qquad a_k < a_{k+1}$$

whose kernel is a periodic function with period T of the form

$$k(t) = \sum_{n=-\infty}^{\infty} K(n) \exp(2\pi i n t / T) \tag{1.2}$$

We consider $K(n)$ to be the values of a function $K(u)$, at n points, which possesses the following properties on the real axis : (*)

1) $K(u)$ is an even, real, continuous function
2) $K(u) > 0, \qquad |u| < \infty$
3) $K(u) = c^2 u^{-2\gamma} [1 + O(u^{-\delta})], \quad (u \to \infty), \quad 0 < \gamma < 1$

$$\delta > 1 - \gamma, \quad (\gamma \leqslant 0.5), \qquad \delta > \gamma \quad (\gamma > 0.5)$$

The spaces $S(\sigma)$, $s(\sigma)$, $C_k^\lambda(a, b)$, $L_p(a, b)$, $C(\gamma)$ introduced in [4] will be used herein, as will also the insignificantly modified spaces H_γ and E. Let us say that $q(x) \in H_\gamma$, if the following inequality holds

(*) If the relations (1) - (3) are satisfied only for discrete values $u = n$, then the function $K(u)$ is constructed nonuniquely by means of the given sequence $K(n)$, by connecting two adjacent points of the n and $n+1$ continuous curve with the properties (1)–(3).

$$\|q\|_H^2 = \sum_{n=-\infty}^{\infty} K(n)\,|Q(\tau n)|^2 < \infty, \quad Q(u) = \sum_{k=1}^{N} \int_{a_{2k-1}}^{a_{2k}} q_{2k-1}(\xi)\, e^{iu\xi} d\xi$$

$$\tau = 2\pi / T$$

Let us consider the function $f(x) \in E$, periodic with period T if the coefficients of its Fourier series possess the property

$$\|f\|_E = \sum_{n=-\infty}^{\infty} |F(n)\,K^{-1}(n)| < \infty, \qquad f(x) = \sum_{n=-\infty}^{\infty} F(n)\, e^{i\tau nx} \tag{1.3}$$

We shall let $f_{2k-1}(x)$ denote values of the function $f(x)$ taken from the segment $[a_{2k-1}, a_{2k}]$. Herein, as in [4], it turns out to be convenient to take values of some function $f(x) \in E$ on the segments $[a_{2k-1}, a_{2k}]$ as the right sides $f_{2k-1}(x)$ in (1.1). The following two theorems are proved by the same method as in [4].

Theorem 1.1. The operator K acts from $L_p(0, T)$ into $C(0, T)$ continuously for $(2\gamma)^{-1} < p < \infty$ $(\gamma \leqslant 0.5)$ and $1 < p < \infty$ $(0.5 < \gamma)$ Let us introduce an operator $\mathbf{M}$ of the form

$$\mathbf{M}q \equiv \sum_{k=1}^{N} \int_{a_{2k-1}}^{a_{2k}} m(x-\xi)\, q_{2k-1}(\xi)\, d\xi, \quad m(x) = \sum_{n=-\infty}^{\infty} M(n)\, e^{i\tau nx} \tag{1.4}$$

Here $M(u)$ is a real, even function continuous on the real axis which possesses the behavior at infinity

$$M(u) = O(u^{-2\gamma-\delta})$$

Theorem 1.2. For values of λ from the circle $|\lambda| < \varkappa^{-1}$ the equation

$$\mathbf{K}q + \lambda\mathbf{M}q = 2\pi f \tag{1.5}$$

cannot have more than one solution in L_p, $p = 2$ $(\gamma \leqslant 0.25)$; $(2\gamma)^{-1} < p \leqslant 2$ $(0.25 < \gamma \leqslant 0.5)$; $1 < p \leqslant 2$ $(0.5 < \gamma)$. if only

$$\varkappa = \|MK^{-1}\|_C < 1 \tag{1.6}$$

Evidently (1.1) $(\lambda = 0)$ is a particular case of (1.5).

2. Let us establish the solvability of (1.1) by the method given in [4]. Utilizing Lemma 3.1 of the paper mentioned, let us represent the function $K(u)$ as

$$K(u) = K_s(u) + M_s(u) \tag{2.1}$$

Here $K_s(u)$ is a meromorphic function in the complex plane which possesses the properties (1) - (3) of Sect. 1, and $M_s(u)$ possesses all the properties of the function $M(u)$ described in Sect. 1, and moreover

$$\|M_s(u)\,K_s^{-1}(u)\|_C \to 0, \qquad s \to \infty \tag{2.2}$$

As in [4], the zeroes and poles of the function $K_s(u)$ which lie in the upper half-plane will be denoted by z_n, ζ_n . respectively.

Let us now turn to an investigation of (1.1) in which $K_s(u)$ plays the part of the function $K(u)$ i.e., the kernel of this equation is

$$k(t) = \sum_{n=-\infty}^{\infty} K_s(n) \exp i\tau nt \tag{2.3}$$

Applying the Poisson summation formula to the series (2.3) and evaluating the integral by using residues, we represent $k(t)$ as

$$k(t) = \sum_{r=1}^{\infty} s_r \left[\exp i\zeta_r\tau|t| + \frac{2\exp 2\pi i\zeta_r}{1-\exp 2\pi i\zeta_r} \cos \zeta_r\tau t\right] \tag{2.4}$$

$$s_r = 2\pi i\,[K_s^{-1}(\zeta_r)]'^{-1}, \qquad |t|<T$$

Let us now seek the solution of the integral equation (1.1) with the right side (1.3) as the series

$$\begin{aligned} q_{2k-1}(x) = \sum_{n=-\infty}^{\infty} F(n) K_s^{-1}(n) \exp in\tau x + \sum_{l=1}^{\infty}\sum_{v=0}^{p(l)} [x_l(2k-1, v) \\ (x-a_{2k-1})^v \exp iz_l\tau(x-a_{2k-1}) + y_l(2k-1, v) \\ (a_{2k}-x)^v \exp iz_l\tau(a_{2k}-x)]\,\tau^v, \qquad a_{2k-1}\leqslant x\leqslant a_{2k} \end{aligned} \tag{2.5}$$

Here again as in [4], $p(l)+1$ is the multiplicity of the zero z_l in the upper half-plane; starting with some sufficiently large number l all $p(l)\equiv 0$. Let us insert (2.4), (2.5) into the left side of (1.1), let us integrate and equate to the right side (1.3). We hence obtain an infinite system to determine the unknown constants X_k and Y_k :

$$\begin{aligned} AX_k + C_kY_k + \sum_{m=1}^{k-1} [B(1, m) X_m + B(2, m) Y_m] + \\ + \sum_{m=1}^{N} [B(3, m) X_m + B(4, m) Y_m] = L_k \end{aligned} \tag{2.6}$$

$$\begin{aligned} AY_k + C_kX_k + \sum_{m=k+1}^{N} [D(1, m) Y_m + D(2, m) X_m] + \\ + \sum_{m=1}^{N} [D(3, m) Y_m + D(4, m) X_m] = G_k \quad (k = 1, 2, \ldots, N, \end{aligned}$$

$$X_0 = Y_0 = X_{N+1} = Y_{N+1} = 0,\ X_k = \{x_l(2k-1, v)\},\ Y_k = \{y_l(2k-1, v)\}$$

The elements of the matrices A, C_k, $B(n, m)$, $D(n, m)$ can be obtained from the appropriate elements of the system (3.8) in [4], in which all the exponents of the exponentials must be multiplied by τ before the operation D is taken. For example, the term $D^v \exp iz_l(a_{2k}-a_{2k-1})$ from [4] must be replaced in our system by

$$D^v \exp i\tau z_l(a_{2k}-a_{2k-1})$$

The elements of the matrices $B(n, m)$, $D(n, m)$ $(n = 3, 4)$ are

$$B(n, m) = \Omega B(n-2, m),\ D(n, m) = \Omega D(n-2, m) \tag{2.7}$$

Here Ω is a diagonal matrix whose elements are

$$\omega_{rr} = (1-\exp 2\pi i\zeta_r)^{-1} \exp 2\pi i\zeta_r \tag{2.8}$$

Let us represent the right sides in (2.6) as

$$L_k = \{l_r(k)\} = \left\{\sum_{m=1}^{k-1} l_r(m) + \sum_{m=1}^{N} \omega_{rr} l_r(m) + c_r(k)\right.$$

$$G_k = \{g_r(k)\} = \left\{ \sum_{m=k+1}^{N} \tau_r(m) + \sum_{m=1}^{N} \omega_{rr}\tau_r(m) + s_r(k) \right\}$$

$$t_r(m) = \sum_{n=-\infty}^{\infty} F(n) K_s^{-1}(n) (\zeta_r - n)^{-1} \{\exp i\tau [\zeta_r (a_{2k-1} - a_{2m-1}) +$$

$$+ na_{2m-1}] - \exp i\tau [\zeta_r (a_{2k-1} - a_{2m}) + na_{2m}]\} \qquad (2.9)$$

$$\tau_r(m) = \sum_{n=-\infty}^{\infty} F(n) K_s^{-1}(n) (\zeta_r - n)^{-1} \{\exp i\tau [\zeta_r (a_{2m} - a_{2k}) +$$

$$+ na_{2m}] - \exp i\tau [\zeta_r (a_{2m-1} - a_{2k}) + na_{2m-1}]\}$$

$$c_r(k) = \sum_{n=-\infty}^{\infty} \frac{F(n) \exp i\tau n a_{2k-1}}{(\zeta_r - n) K_s(n)}, \quad s_r(k) = \sum_{n=-\infty}^{\infty} \frac{F(n) \exp i\tau\, n a_{2k}}{(\zeta_r + n) K_s(n)}$$

The following is valid.

Lemma 2.1. The operator $A^{-1}\,\Omega R$ is continuous from any $S(\sigma)$, $\sigma > 0$ into $S(1-\gamma)$. Here R is any of the operators C, B, D. The elements of the matrix A^{-1} are presented in [4].

By virtue of Lemma 2.1 the infinite system (2.6) is no different from the system (3.8) investigated in [4] in its functional properties. But then all the reasoning in Sect. 3 of the paper mentioned can be applied by first considering (1.1) with the kernel (2.4) and the right side (3.9), (3.10) from [4].

Theorem 2.1. Let $f(x) \in E$. Then the system (2.6) is uniquely solvable in $S(1-\gamma)$ and (1.1) with the kernel (2.4) is solvable in $C(\gamma)$. The following estimate hence holds

$$\| q \|_{C(\gamma)} \leqslant \| \mathbf{K}_s^{-1} \| \, \| f \|_E \qquad (2.10)$$

To prove the solvability of (1.1) with a general kernel, let us examine (1.5) in which the functions $K_s(u)$ and $M_s(u)$ constructed above are taken as $K(u)$ and $M(u)$. For $\lambda = 1$ equation (1.5) agrees with (1.1).

Lemma 2.2. The operator $\mathbf{K}_s^{-1}\mathbf{M}_s$ is completely continuous in $C(\sigma)$, $\gamma \leqslant \sigma < \varkappa^\circ$ ($\varkappa^\circ = \inf(\delta, 2\gamma)$, $\gamma < 0.5$; $\varkappa^\circ = \inf(\delta, 1)$, $\gamma \geqslant 0.5$). Combining Theorem 1.2 and Lemma 2.2. and representing (1.1) as

$$q + \lambda K_s^{-1} M_s q = K_s^{-1} f \qquad (2.11)$$

we conclude that the method of successive approximations by means of the formula

$$q_{m+1} = -\lambda \mathbf{K}_s^{-1} \mathbf{M}_s q_m \qquad (2.12)$$

will converge for all $|\lambda| < \varkappa^{-1}$.

Theorem 2.2. Equation (1.1) is uniquely solvable for all $f(x)$ such that

$$\mathbf{K}_s^{-1} f \in C(\gamma) \qquad (2.13)$$

The solution is representable as

$$q = \sum_{m=0}^{\infty} (-1)^m (\mathbf{K}_s^{-1}\mathbf{M}_s)^m \mathbf{K}_s^{-1} f \qquad (2.14)$$

$$\| q \|_{C(\gamma)} \leqslant C \| \mathbf{K}_s^{-1} f \|_{C(\gamma)}, \qquad C = \sum_{m=0}^{\infty} \| (\mathbf{K}_s^{-1}\mathbf{M}_s)^m \| \qquad (2.15)$$

The norm of the operator $\mathbf{K}_s^{-1}\mathbf{M}_s$ is defined for its operation in $C(\gamma)$. Let us note that (2.13) holds if $f \in E$. In this case (2.15) becomes

$$\|q\|_{C(\gamma)} \leqslant \|\mathbf{K}^{-1}\| \|f\|_E, \quad \|\mathbf{K}^{-1}\| = C \|\mathbf{K}_s^{-1}\|_{E \to C(\gamma)} \tag{2.16}$$

3. Let us apply the results obtained to give a foundation to the approximate factorization [4]. Let us assume that two integral equations are given

$$\mathbf{K}_1 q_1 = 2\pi f, \qquad \mathbf{K}_2 q_2 = 2\pi f$$

in which the Fourier transforms of the kernels $K_1(u)$ and $K_2(u)$ satisfy the conditions (1)–(3) of Sect. 1, and $f \in E$.

The question arises for which proximity relationship between $K_1(u)$ and $K_2(u)$ will proximity of the solutions q_1 and q_2 of the equations presented hold in $C(\gamma)$. An answer would be to recommend for the selection of the function $K_2(u)$ an easily factorizable function approximating $K_2(u)$ and assuring proximity of the approximate solution q_1 to the exact q_2 in the metric of $C(\gamma)$.

Theorem 3.1. Let the quantity

$$\varepsilon = \sup_n |K_1(n) - K_2(n)| K_1^{-1}(n)(1 + |n|)^\alpha \quad (n = 0, \pm 1, \pm 2, \ldots)$$

$$\alpha > 1 - \gamma \quad (\gamma < 0.5), \qquad \alpha > \gamma \quad (0.5 \leqslant \gamma)$$

satisfy the condition

$$\varepsilon < \|\mathbf{K}_1^{-1}\|^{-1} L^{-1}$$

Then the following estimate holds

$$\|q_2 - q_1\|_{C(\gamma)} \leqslant \varepsilon L \|\mathbf{K}_1^{-1}\| (1 - \varepsilon L \|\mathbf{K}_1^{-1}\|)^{-1} \|q_1\|_{C(\gamma)}$$

The expression for L has been presented in [4] and $\|\mathbf{K}_1^{-1}\|$ is given by the relation (2.16). The theorem is proved by the same method as Theorem 5.1 in [4].

4. Let us examine the case $N = 1$ in more detail. The infinite system to determine the coefficients $X_1 = \{x_l(1, v)\}$, $Y_1 = \{y_l(1, v)\}$ becomes

$$AX_1 + C_1 Y_1 + B(3,1) X_1 + B(4,1) Y_1 = L_1 \tag{4.1}$$

$$AY_1 + C_1 X_1 + D(3,1) Y_1 + D(4,1) X_1 = G_1$$

Putting $v = 0$ in the system (4.1) (all the zeroes of the function $K_s(z)$ are single), and $F(n) = 1$, $F(k) = 0$ $(k \neq n)$, we obtain a system of the form

$$\sum_{l=1}^{\infty} \left\{ \frac{1 - \exp i\tau [z_l\sigma + \zeta_r (T - \sigma)]}{\zeta_r - z_l} \pm \frac{\exp i\tau z_l \sigma - \exp i\tau\zeta_r (T - \sigma)}{\zeta_r + z_l} \right\} x_l^{\pm} =$$

$$= \frac{\exp i\tau n a_1 - \exp i\tau [a_2 n + \zeta_r (T - \sigma)]}{(n - \zeta_r) K(n)} \mp \frac{\exp i\tau n a_2 - \exp i\tau [a_1 n + \zeta_r (T - \sigma)]}{(n + \zeta_r) K(n)} \tag{4.2}$$

Here

$$\{x_l^{\pm}\} = X^{\pm} = X_1 \pm Y_1, \qquad \sigma = a_2 - a_1 < T$$

Taking into account the invertibility of the matrix A, we conclude that the system (4.2) is the most specified (the matrix A has the least perturbation) for the relation $2\sigma = T$ between the parameters. The exponential functions of the coefficients in the system (4.2) have negative exponents of the same order in this case, under the condition that the zeroes z_l and poles ζ_r are of the same order. In case $\zeta_r \gg z_1$, (4.2) will evidently be close to the system (2.19) investigated in [5]. This latter means that the solution of the integral equation (1.1) is close to the solution of the integral equation

$$K_q = \int_{-a}^{a} k(x-\xi)\, q^*(\xi)\, d\xi = 2\pi f(x), \quad |x| \leqslant a, \quad \tau a_1 = -a, \quad \tau a_2 = a \tag{4.3}$$

$$k(t) = \int_{-\infty}^{\infty} K_s(u)\, e^{iut}\, du, \quad f(x) = \int_{-\infty}^{\infty} F(\eta)\, e^{i\eta x} d\eta, \quad q^*(\xi) = \tau q(\xi/\tau)$$

Equation (4.3) has been investigated in [4], and in [6, 7] for large a.

The result presented in these papers permits estimation of the norm of the inverse operator of (4.3) which is always necessary to estimate the accuracy of the approximate solution.

5. Presented below is one of the procedures permitting estimation of the norm of the inverse operator of the mentioned equation for large a. According to [6], for

$$f(x) = \exp i\eta x \qquad (\Phi(\xi) = \delta(\xi - \eta))$$

the solution of (4.3) is

$$q_\eta(x) = K^{-1}(\eta)\, e^{i\eta x} - \sum_{k=0}^{\infty} (-1)^k \{ \mathrm{S}(a+x)\, \mathbf{F}^k(a, z)$$

$$\psi[t, \eta(-1)^k] + \mathrm{S}(a-x)\, \mathbf{F}^k(a, z)\, \psi[t, \eta(-1)^{k+1}] \} \tag{5.1}$$

Here the operators $\mathbf{F}(a, z)$, $\mathrm{S}(x)$ are given, respectively, by (7) and (13) from the paper mentioned. The operator $\mathbf{F}(a, z)$ is continuous in the space A of functions regular in the lower half-plane and bounded with weight z there. The norm of the operator $\mathbf{F}(a, z)$ in the space A is given by the right side of the relation (10) in [6].

Lemma 5.1. The operator $\mathrm{S}(a \pm x)$ is continuous from A into $C(0.5)$ [4].

It is necessary to show that

$$\max_{|x| \leqslant a} |(a^2 - x^2)^{0.5}\, \mathrm{S}(a \pm x) f| \leqslant M \|f\|_A \tag{5.2}$$

If $f \in A$, then the operator $S(x)$ can be represented, on the basis of (13), as (the contour in the lower half-plane is deformed)

$$\mathrm{S}(x) f = \frac{1}{2\pi i} \int_{\gamma} K_+^{-1}(t)\, f(t)\, e^{-itx} dt$$

Let us assume

$$m = \inf_\gamma \max_{t \in \gamma} |K_+^{-1}(t)\, t^{-0.5}| \tag{5.3}$$

Under the assumptions in [6] $m < \infty$. Then

$$|x^{0.5}\, \mathrm{S}(x) f| \leqslant \frac{x^{0.5}}{2\pi} \int_{\gamma} \left| \frac{f(t)\, e^{-itx}}{K_+(t)} \right| |dt| \leqslant \frac{m x^{0.5}}{2\pi} \int_{\gamma} \left| \frac{e^{-itx}}{t^{0.5}} \right| |dt|\, \|f\|_A \leqslant$$

$$\leqslant \frac{m}{2\pi} \int_{\gamma x} \left| \frac{e^{-it}}{t^{0.5}} \right| |dt|\, \|f\|_A$$

Using the notation

$$\|\mathrm{S}\| \leqslant M = \frac{m}{2\pi} \max_x \int_{\gamma x} \left| \frac{e^{-it}}{t^{0.5}} \right| |dt| < \infty, \quad x \in [0, 2a] \tag{5.4}$$

we obtain (5.2). The lemma is proved.

Lemma 5.2. Let $f(x) \in C_1^\lambda(-a, a)$ $(\lambda > 0)$. Then there exists its continuation $f^*(x) \in C_1^\lambda(-b, b)$ in $[-b, b] \supset [-a, a]$ such that its Fourier transform $\Phi(\eta)$ satisfies the relation

$$|\Phi(\eta)|(1 + |\eta|)^{1+\lambda} < C\|f\|_{C_1^\lambda(-a,a)} \tag{5.5}$$

Lemma 5.2 is proved by methods elucidated in [8]. The inequality

$$\|\psi_k\|_A < \theta\|f\|_{C_1^\lambda(-a,a)} \qquad (\lambda > 0.5) \tag{5.6}$$

is proved on the basis of Lemma 5.2.

Lemma 5.3. For $f \in C_1^\lambda(-a, a)$ the estimate

$$\left\| \int_{-\infty}^{\infty} \frac{\Phi(\eta)}{K(\eta)} e^{i\eta x} d\eta \right\|_{C(-a,a)} \leqslant N\|f\|_{C_1^\lambda(-a,a)} \tag{5.7}$$

is valid. To prove Lemma 5.3 it is sufficient to utilize the customary procedure of inverting the Fourier transform [8] and the property (2) in [6].

Combining Lemmas 5.1, 5.3 and the inequality (5.6), we obtain

$$\|q\|_{C(0.5)} \leqslant [N + 2\|\mathbf{S}\|(1 - \|\mathbf{F}\|)^{-1}\theta]\|f\|_{C_1^\lambda(-a,a)}$$

Therefore

$$\|\mathbf{K}^{-1}\| \leqslant N + 2\|\mathbf{S}\|(1 - \|\mathbf{F}\|)^{-1}\theta \tag{5.8}$$

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